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Best Uniform Approximation with Hermite–Birkhoff Interpolatory Side Conditions

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Best approximation to continuous functions by polynomials satisfying Hermite–Birkhoff interpolation conditions is discussed. Characterization, sufficient conditions for uniqueness, and the alternation property of these polynomials are studied. The results obtained extend work on best approximation with interpolatory side conditions of Hermite type. By this extension the space of polynomials that plays a role in the approximation is no longer a Haar space, and the results depend strongly on the structure of the side conditions.

INTRODUCTION

The purpose of this paper is to characterize best uniform approximation to a function $f \in C[a, b]$ by polynomials of degree $\leqslant n - 1$ satisfying $r < n$ interpolatory side conditions of Hermite–Birkhoff (HB) type. Special cases of this problem were studied in [4, 11, 14, 15]. In [4, 14, and 15], the interpolatory conditions are imposed only on the values of the approximating polynomials at $r < n$ points (Lagrange interpolation). An extension of these results to Hermite interpolatory constraints is carried out in [11].

In Section 1 we introduce the HB interpolation problem in terms of incidence matrices and summarize some known results on this problem, that are relevant to our work. We extend these results to the case in which the number of interpolatory conditions r is less than n.

The results in the following sections are formulated in terms of the structure and properties of the incidence matrix describing the interpolatory conditions. Section 2 contains the formulation of the main problem, a generalization of the Kolmogorov Theorem and conditions for uniqueness of the best approximating polynomial. In Section 3 we prove that a polynomial of best approximation satisfies a weak alternation property. For a limited class of incidence matrices (when there is also uniqueness) this property of a polynomial is also a sufficient condition for best approximation. The proof of the alternation property relies on the general result of Lemma 3.1, which gives a sufficient condition under which a certain system of functions is a Haar system.

Most of the theorems in this work are sharp, as is demonstrated by several examples. Other examples are presented to show that new definitions or requirements are of significance.

The results in this paper also apply to the case of best approximation by an extended Tchebycheff system under interpolatory constraints if differentiation is replaced by suitable differential operators [8].

1. Hermite–Birkhoff Interpolation

The customary formulation of Hermite–Birkhoff interpolation problem is stated in terms of a $k \times n$ incidence matrix,

$$E_n^k = (e_{ij}) \qquad i = 1,\ldots, k; \quad j = 0, 1,\ldots, n-1.$$

Each e_{ij} is 0 or 1 and $\sum_{ij} e_{ij} = n$. Given k real points,

$$\xi_1 < \xi_2 < \cdots < \xi_k, \tag{1.1}$$

the matrix E_n^k describes the problem of finding a polynomial $p(x) \in \pi_{n-1}$ (π_{n-1} is the class of polynomials of degree $\leqslant n-1$) that satisfies

$$p^{(j)}(\xi_i) = b_{ij} \qquad (e_{ij} = 1) \tag{1.2}$$

for any n given constants b_{ij} ($e_{ij} = 1$). The matrix E_n^k is said to be order-poised if (1.2) has a unique solution for every choice of the ordered points (1.1) or, equivalently, if the homogeneous problem [(1.2) with all $b_{ij} = 0$] has only the trivial solution. Thus, E_n^k is order-poised if and only if the determinant of the linear system (1.2), denoted by $\Delta(E_n^k, \bar{\xi})$, is nonzero for all $\bar{\xi} = (\xi_1, \ldots, \xi_k)$ satisfying (1.1). For a survey on the characterization of order-poised matrices, refer to [7, 19]. Here we mention only some results that will be used later.

A necessary condition for order-poisedness is given by [18]:

Theorem A. *Let $m_j = \sum_{i=1}^{k} e_{ij}$ and $M_j = \sum_{i=0}^{j} m_i$ ($j = 0, 1,\ldots, n-1$) be the Polya constants. Then, if E_n^k is order-poised, it satisfies the Polya conditions*

$$M_j \geqslant j + 1 \qquad j = 0, 1,\ldots, n-1. \tag{1.3}$$

Remark 1.1. For two-point interpolation problems ($k = 2$) the Polya conditions (1.3) are also sufficient for order poisedness [19].

Remark 1.2. If the Polya conditions fail, there is no set of points (1.1) for which (1.2) has a unique solution [19].

In the following we deal with incidence matrices for which there is at least one set of points (1.1) such that (1.2) has a unique solution. By Remark 1.2 all such matrices satisfy the Polya conditions. In this case, the set of points (1.1) for which the homogeneous problem has a nontrivial solution is characterized by the following theorem:

THEOREM B. *If E_n^k satisfies the Polya conditions* (1.3), *then the set of ordered k-tuples satisfying $\Delta(E_n^k, \bar{\xi}) = 0$ is a closed set with an empty interior in R^k.*

This result follows easily from similar results proved by Ferguson for the complex case [6].

In this work we deal with classes of polynomials that satisfy HB interpolation conditions described by incidence matrices with $r \leqslant n$ nonzero entries.

Let $E_n^k(r) = (e_{ij})$ denote a $k \times n$ incidence matrix for which $\sum_{i,j} e_{ij} = r \leqslant n$, $\bar{\xi} = (\xi_1, \xi_2, \ldots, \xi_k)$ a given ordered k-tuple, and let

$$P_0 = P_0(E_n^k(r), \bar{\xi}) = \{p \mid p \in \pi_{n-1}, p^{(j)}(\xi_i) = 0, e_{ij} = 1\}. \tag{1.4}$$

Obviously P_0 is a linear space of dimension $\geqslant n - r$. Any matrix $E_n^k(r)$ for which $P_0(E_n^k(r), \bar{\xi})$ is of dimension $n - r$ for every choice of $\bar{\xi}$ can be completed into an order-poised matrix by adding to it $n - r$ units, some of which may occur in new rows.

The following results are direct consequences of Theorems A and B and Remarks 1.1 and 1.2.

Remark 1.3. The generalized form of Polya conditions

$$M_j \geqslant j + 1 - (n - r) \qquad j = 0, 1, \ldots, n - 1 \tag{1.5}$$

are necessary for $P_0(E_n^k(r), \bar{\xi})$ to be of dimension $n - r$ for every choice of $\bar{\xi}$. If $k = 2$, then conditions (1.5) are also sufficient.

Remark 1.4. The dimension of $P_0(E_n^k(r), \bar{\xi})$ is $n - r$ for at least one choice of $\bar{\xi}$, if and only if, conditions (1.5) hold.

DEFINITION 2.2. A Hermitian block of length μ in an incidence matrix is a sequence of consecutive 1's in one of its rows, beginning in the first column: $e_{ij} = 1, j = 0, 1, \ldots, \mu - 1$, $e_{i\mu} = 0$.

Given an incidence matrix $E_n^k(r)$ at $\bar{\xi}$, we are interested in two types of additional interpolatory conditions:

DEFINITION 2.3. An L-condition (L for Lagrange) is a condition corresponding to an additional unit in the first column of $E_n^k(r)$, (possibly in a new row). An H-condition (H for Hermite) is a condition corresponding to an additional unit at the end of a Hermitian block of $E_n^k(r)$.

THEOREM 1.1. *Let $E_n^k(r)$ satisfy conditions* (1.5), *then by adding to it $n - r$ units corresponding to* L-*conditions and/or* H-*conditions one at a row, we get a matrix satisfying the Polya conditions* (1.3).

Proof. Let μ^* be the length of the longest Hermitian block of $E_n^k(r)$. Then, all additional $n - r$ units are in columns up to μ^*. By conditions (1.5), for all $j \geqslant \mu^*$,

$$M_j \geqslant j + 1 - (n - r) + (n - r) = j + 1,$$

while for all $j < \mu^*$, $M_j \geqslant j + 1$ since there exists at least one Hermitian block of length μ^*.

2. CHARACTERIZATION AND UNIQUENESS OF BEST APPROXIMATION

Given a function $f \in C[a, b]$, a $k \times n$ incidence matrix $E_n^k(r) = (e_{ij})$, k fixed points $a \leqslant \xi_1 < \xi_2 < \cdots < \xi_k \leqslant b$, and $r - m_0$ $(m_0 = \sum_{i=1}^k e_{i0})$ fixed numbers $\{b_{ij} \mid e_{ij} = 1, j \geqslant 1\}$, we define the class:

$$P = P(E_n^k(r), \xi) = \{p \mid p \in \pi_{n-1}, p(\xi_i) = f(\xi_i) \text{ for } e_{i0} = 1$$
$$\text{and } p^{(j)}(\xi_i) = b_{ij} \text{ for } e_{ij} = 1, j \geqslant 1\}. \tag{2.1}$$

We assume that P_0, defined in (1.4), is of dimension $n - r$ or, equivalently, that the r conditions prescribed by $E_n^k(r)$ at ξ are linearly independent.

In the following we characterize the polynomials of best approximation (pba) to $f \notin P$ by polynomials of the class P in the uniform norm

$$\| f - p \| = \max_{a \leqslant x \leqslant b} | f(x) - p(x)|.$$

Compactness arguments show that if P is not empty then a pba to f from P exists.

THEOREM 2.1. *$p \in P$ is a* pba *to a function $f \in C[a, b]$, if and only if, each polynomial $p_0(x) \in P_0(E_n^k(r), \xi)$ satisfies*:

$$\max_{x \in A} [f(x) - p(x)]\, p_0(x) \geqslant 0$$

where

$$A = A(f, p) = \{x \mid a \leqslant x \leqslant b, |f(x) - p(x)| = \|f - p\|\} \qquad (2.2)$$

We omit the proof which is similar to the proof of the well-known Kolmogorov Theorem [12].

Further results can be formulated in terms of the following property of the incidence matrix $E_n^k(r)$:

DEFINITION 2.1. A $k \times n$ incidence matrix $E_n^k(r)$ is called L-poised at a fixed point $\bar{\xi}$, with respect to the interval $[a, b]$, (or shortly L-poised), if by any addition of $n - r$ L-conditions (Definition 1.2) in $[a, b]$ the resulting matrix describes an interpolation problem with a unique solution.

The following example is presented to show that the linear independence of the conditions prescribed by a matrix does not guarantee its L-poisedness:

EXAMPLE 2.1. For $E = E_3^2(2) = \left(\begin{smallmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{smallmatrix}\right)$, $P_0(E, \bar{\xi})$ is of dimension $n - r = 3 - 2 = 1$ for every choice of $\bar{\xi}$, since $E_3^2 = \left(\begin{smallmatrix}1 & 1 & 0\\ 0 & 1 & 0\end{smallmatrix}\right)$ is order poised. But it is not L-poised at $\bar{\xi} = (\xi_1, \xi_2)$ with respect to the interval $[a, b]$, $a \leqslant \xi_1$, $b \geqslant 2\xi_2 - \xi_1$, since for the matrix

$$E_3^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$\Delta(E_3^3, \bar{\xi}) = 0$ for $\bar{\xi} = (\xi_1, \xi_2, 2\xi_2 - \xi_1)$.

The characterization of L-poised matrices is inherently connected to the problem of characterizing poised matrices, which is still open. A partial answer to the latter problem yields a broad class of L-poised matrices that are matrices satisfying (1.5) with all blocks either Hermitian or composed of even numbers of units [1]. (A block or a sequence in an incidence matrix is a string of 1's in one of its rows). A full characterization of L-poisedness of one row matrices can be concluded from the results in [9].

For L-poised matrices the pba to f from P has similar properties to those of the general pba (no side conditions). Weaker results can still be obtained for any matrix $E_n^k(r)$ by considering matrices composed of its first $\bar{n}$ columns. Although the conditions prescribed by the units in these $\bar{n}$ columns are linearly independent on π_{n-1}, they might be dependent when imposed on polynomials of degree $\leqslant \bar{n} - 1$. Thus we introduce the following concept:

DEFINITION 2.2. An incidence matrix $\bar{E} = E_{\bar{n}}^k(\bar{r})$ is called a partial matrix of $E_n^k(r)$ at $\bar{\xi}$ if it is composed of the first $\bar{n}$ columns of $E_n^k(r)$, from which all units corresponding to linearly dependent conditions on $\pi_{\bar{n}-1}$ have been omitted.

Remark 2.1. From the above definition it is obvious that for any partial matrix $E_{\bar{n}}^k(\bar{r})$ of $E_n^k(r)$ at $\bar{\xi}$, $\bar{r} \leqslant M_{\bar{n}-1}$ and

$$P_0(E_{\bar{n}}^k(\bar{r}), \bar{\xi}) = \pi_{\bar{n}-1} \cap P_0(E_n^k(r), \xi). \tag{2.3}$$

$$\bar{n} - \bar{r} \leqslant n - r. \tag{2.4}$$

To illustrate Definition 2.2 we bring the following:

EXAMPLE 2.2. Let

$$E = E_4^3(3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and let $\bar{\xi} = (-1, 0, 1)$. The conditions prescribed by E at $\bar{\xi}$ are linearly independent on π_3. For $\bar{n} = 3$ a partial matrix of E is

$$E_3^3(2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where the units in the second column have been deleted since for all $p \in P_0(E_3^3(2), \bar{\xi})$, $p'(0) = 0$.

LEMMA 2.1. *Every incidence matrix $E_n^k(r)$ has at least one partial matrix at $\bar{\xi}$ which is* L*-poised.*

Proof. Let $\sigma = \sigma(E_n^k(r))$ denote the sum of the lengths of all Hermitian blocks (H-blocks) of $E_n^k(r)$ ($\sigma = 0$ if there is no H-block). If $\sigma = 0$ then the partial matrix with $\bar{n} = 1$, $\bar{r} = 0$ is L-poised.

If $\sigma \geqslant 1$ then the partial matrix with $\bar{n} = \sigma$, $\bar{r} = \sigma$, which contains only the set of all H-blocks of $E_n^k(r)$, is poised and therefore L-poised at $\bar{\xi}$.

The above result and definitions enable us to present:

THEOREM 2.2. *Let $P^* = P^*(E_n^k(r), \bar{\xi}, f) \subset P(E_n^k(r), \bar{\xi})$ be the set of all* pba *to $f \in C[a, b]$ from P, and let $\bar{n}$ be the maximal integer such that the partial matrix $E_{\bar{n}}^k(\bar{r})$ of $E_n^k(r)$ is* L*-poised. Then the set $A^* = \bigcap_{p \in P^*} A(f, p)$, where $A(f, p)$ is defined in* (2.2), *contains at least $\bar{n} + 1 - \bar{r}$ points.*

Proof. First we prove that each $A = A(f, p)$, $p \in P^*$, contains at least $\bar{n} + 1 - \bar{r}$ points. If $\bar{n} = \bar{r}$ then the claim is obvious by continuity of $f - p$. Thus suppose that for some $p \in P^*$, $A = A(f, p) = \{x_1, ..., x_s\}$, where $1 \leqslant s \leqslant \bar{n} - \bar{r}$, $\bar{n} > \bar{r}$. Since A is finite

$$\max_{a \leqslant x \leqslant b} |f(x) - p(x)| = \|f - p\| > 0.$$

There exists a nontrivial polynomial $p_0(x)$ of degree at most $\bar{n}-1$ satisfying the homogeneous conditions prescribed by $E_{\bar{n}}^k(\bar{r})$ at $\bar{\xi}$ together with:

$$p_0(x_m) = -[f(x_m) - p(x_m)], \qquad m = 1, 2, \ldots, s. \tag{2.5}$$

This follows since $E_{\bar{n}}^k(\bar{r})$ is L-poised at $\bar{\xi}$ and the number of imposed conditions is only $\bar{r} + s \leqslant \bar{r} + \bar{n} - \bar{r} = \bar{n}$. (No point $x_i \in A$ can coincide with a point ξ_m for which there is $e_{m0} = 1$ in $E_n^k(r)$ since at such a point $f(\xi_m) = p(\xi_m)$, while x_i is a point where $|f(x) - p(x)|$ attains its positive maximum.)

By (2.3) $p_0 \in P_0(E_n^k(r), \xi)$, and by (2.5) p_0 is a nontrivial polynomial for which:

$$\max_{x \in A} [f(x) - p(x)]\, p_0(x) = \max_{1 \leqslant i \leqslant s} [f(x_i) - p(x_i)]\, p_0(x_i) = -\,|f - p\,|^2 < 0$$

in contradiction to Theorem 2.1. Thus $A(f, p)$ contains at least $\bar{n} + 1 - \bar{r}$ points for each $p \in P^*$.

Since the set P^* is a convex set, one can find a polynomial $p(x) \in P^*$ such that for every $p_1(x) \in P^*$ there exists $p_2(x) \in P^*$ for which:

$$p(x) = \alpha p_1(x) + \beta p_2(x) \qquad \alpha, \beta > 0 \qquad \alpha + \beta = 1$$

(see [5, p. 16]). By the triangle inequality, for any $x_i \in A(f, p)$:

$$\epsilon = \|f - p\| = |f(x_i) - p_1(x_i)| = |f(x_i) - p_2(x_i)| \tag{2.6}$$

$$p_1(x_i) = p_2(x_i) = p(x_i) \tag{2.7}$$

Hence, $A(f, p) \subset A(f, p_1)$ for all $p_1 \in P^*$. Since $A(f, p)$ contains at least $\bar{n} + 1 - \bar{r}$ points, so does A^*.

Remark 2.2. From the proof it is evident that the same result holds for any partial matrix of E that is L-poised at $\bar{\xi}$, but (2.4) indicates that by taking a smaller L-poised partial matrix of $E_n^k(r)$, we may get a weaker result.

The following lemma deals with a class of incidence matrices for which Theorem 2.2 gives the weakest possible result, i.e., it guarantees only one point in $A(f, p)$ for all $p \in P^*$.

LEMMA 2.2. *If all $p_0 \in P_0(E_n^k(r), \bar{\xi})$ have a common zero $y \in [a, b]$ that is not prescribed by $E_n^k(r)$ at $\bar{\xi}$, then for any* L-*poised partial matrix $E_{\bar{n}}^k(\bar{r})$, $\bar{n} = \bar{r}$.*

Proof. Assume to the contrary, that there exists a partial matrix $E_{\bar{n}}^k(\bar{r})$ with $\bar{n} > \bar{r}$ that is L-poised. The homogeneous problem described by $E_{\bar{n}}^k(\bar{r})$

at $\bar{\xi}$ with additional $\bar{n} - \bar{r} - 1$ L-conditions in $[a, b] - \{\xi_1, ..., \xi_k, y\}$ has at least one nontrivial solution $\tilde{p}(x)$, where $\tilde{p}(x) \in P_0(E_{\bar{n}}^k(\bar{r}), \bar{\xi})$. By (2.3) $\tilde{p}(y) = 0$; thus, it has $\bar{n} - \bar{r}$ zeroes in $[a, b] - \{\xi_1, ..., \xi_k\}$ in contradiction to the L-poisedness of $E_{\bar{n}}^k(\bar{r})$ at $\bar{\xi}$.

It is easy to construct examples that show that $\bar{n} = \bar{r}$ may occur in Theorem 2.2, even if the only common zeroes of all $p \in P_0(E_n^k(r), \bar{\xi})$ are those prescribed by $E_n^k(r)$ at $\bar{\xi}$.

Example 2.3 presents a matrix from the class of incidence matrices dealt with in Lemma 2.2. This example shows that the number $\bar{n} + 1 - \bar{r}$ in Theorem 2.2 cannot be improved.

The next following theorems and corollaries deal with the unicity problem.

THEOREM 2.3. *Let $E_n^k(r)$ be given, $\bar{n}$, $\bar{r}$ defined as in Theorem 2.2. If $p_1(x)$ and $p_2(x)$ are two distinct* pba *to f from $P(E_n^k(r), \bar{\xi})$, then $p_1(x) - p_2(x)$ is of degree $\geqslant \bar{n}$.*

Proof. According to Theorem 2.2 there exist at least $\bar{n} + 1 - \bar{r}$ points in $A(f, p_1) \cap A(f, p_2)$, where by (2.7):

$$p_1(x_i) - p_2(x_i) = 0 \qquad i = 1, ..., s, \quad s \geqslant \bar{n} + 1 - \bar{r}. \tag{2.8}$$

The polynomial $p_1(x) - p_2(x)$ satisfies, together with (2.8), the $\bar{r}$ conditions prescribed by $E_n^k(r)$ at $\bar{\xi}$, and since $E_{\bar{n}}^k(\bar{r})$ is L-poised and $p_1 - p_2 \not\equiv 0$, the degree of $p_1 - p_2$ is $\geqslant \bar{n}$.

COROLLARY 2.1. *Among all* pba *to f from P there exists at most one of degree $\leqslant \bar{n} - 1$.*

COROLLARY 2.2. *If $E_n^k(r)$ is* L*-poised then there is a unique* pba *to f from P.*

If $E_n^k(r)$ is not L-poised, a sufficient condition for uniqueness is given by:

THEOREM 2.4. *The polynomial $p(x) \in P^*(E_n^k(r), \bar{\xi})$ is unique if each $p_0(x) \in P_0(E_n^k(r), \bar{\xi})$ has less than $\bar{n} + 1 - \bar{r}$ zeroes at points of $A(f, p)$.*

Proof. Suppose $p_1(x) \neq p(x)$ is also a pba to f from P. By Theorem 2.2 there exist at least $\bar{n} + 1 - \bar{r}$ points in $A(f, p) \cap A(f, p_1)$, and by (2.7) in these points

$$p_1(x_i) - p(x_i) = 0, \qquad i = 1, 2, ..., \bar{n} + 1 - \bar{r}.$$

Since $p_1(x) - p_2(x) \in P_0(E_n^k(r), \bar{\xi})$, we get a contradiction to the assumption of the theorem.

For differentiable functions we have a stronger result:

THEOREM 2.5. *The polynomial* $p(x) \in P^*(E_n^k(r), \bar{\xi})$ *is the unique* pba *to* $f \in C^1[a, b]$, *if each* $p_0(x) \in P_0(E_n^k(r), \bar{\xi})$ *has less than* $\bar{n} + 1 - \bar{r}$ *zeroes, which are either at* $\{a, b\}$ *or are of multiplicity* $\geqslant 2$, *at points of* $A(f, p)$.

Proof. As in the proof of the previous theorem if $p_1 \neq p$ is also a pba there exist $\bar{n} + 1 - \bar{r}$ extremal points of both $f(x) - p(x)$ and $f(x) - p_1(x)$, where

$$p(x_i) - p_1(x_i) = 0, \qquad i = 1, 2, \ldots, \bar{n} + 1 - \bar{r}$$

and

$$p'(x_i) - p_1'(x_i) = 0, \qquad x_i \in (a, b) \qquad 1 \leqslant i \leqslant \bar{n} + 1 - \bar{r},$$

in contradiction to the assumption of the theorem.

For functions with more derivatives, the above result can be further improved. We conclude this section by an example demonstrating the sharpness of the above theorems.

EXAMPLE 2.3. Let $f(x) = -\frac{3}{2}x^6 + \frac{3}{2}x^4 + x^3$, $[a, b] = [-1, 1.1]$

$$E = E_5^2(4) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}, \qquad \bar{\xi} = (-1, 0), \qquad b_{ij} = 0 \qquad j \geqslant 1.$$

Then

$$\begin{aligned} P &= P(E, \bar{\xi}) \\ &= \{p \mid p \in \pi_4, p(-1) = f(-1) = -1, p'(-1) = p'(0) = p'''(0) = 0\}. \end{aligned}$$

Each $p \in P$ is of the form $p(x) = a(x^2 - 1)^2 - 1$. $E_5^2(4)$ is not L-poised at $\bar{\xi}$ with respect to the interval $[-1, 1.1]$ since

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is not poised at $(-1, 0, 1)$. For all $p \in P$, $f(1) - p(1) = 2$, therefore any $p \in P$ for which $\| f - p \| = 2$ is a pba (see Fig. 1). It is easily seen that $p(x) \equiv -1$ is a pba and so is every polynomial $p(x) = a(x^2 - 1)^2 - 1$ where $0 \leqslant a \leqslant 3$.

This example shows that if $E_n^k(r)$ is not L-poised and the assumptions of Theorems 2.4 and 2.5 fail, we cannot have uniqueness in general. Here $\bar{n} = \bar{r} = 4$, and according to Theorem 2.3 we have exactly one pba of degree $\leqslant \bar{n} - 1$ ($p(x) \equiv -1$). For this polynomial $| f(x) - p(x)|$ attains its maximal value only once ($\bar{n} + 1 - \bar{r} = 1$) at the only point of A^*, which shows that

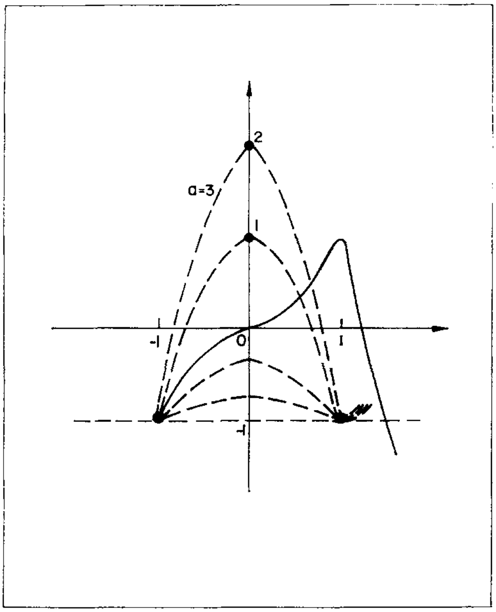

FIGURE 1.

Theorems 2.2, 2.4, and 2.5 are sharp, in the sense that the number $\bar{n} + 1 - \bar{r}$ cannot be increased.

It is easily shown (in a manner similar to the proof in [8, p. 284]) that for every given matrix $E_n^k(r)$ that is not L-poised at $\bar{\xi}$ with respect to $[a, b]$, it is possible to construct a function $f \in C[a, b]$ that satisfies $f(\xi_i) = 0$, $e_{i0} = 0$, and has infinitely many pba's from $P_0(E_n^k(r), \bar{\xi})$.

Corollary 2.2 and the last remark show that L-poisedness at $\bar{\xi}$ of $E_n^k(r)$ is a necessary and sufficient condition for uniqueness of pba from $P_0(E_n^k(r), \bar{\xi})$ to continuous functions which vanish on the set:

$$S = \{\xi_i \mid 1 \leqslant i \leqslant k, \mu_i > 0\} \tag{2.9}$$

where

$$\mu_i = \min\{j \mid e_{ij} = 0, 0 \leqslant j \leqslant n - 1\}, \qquad i = 1, 2,..., k \tag{2.10}$$

This condition is weaker than the Haar condition because it states that any nontrivial polynomial in the $n - r$ dimensional subspace P_0 can have at most $n - r - 1$ zeroes in $[a, b] - S$ (or at most $n - r - 1 + m_0$ zeroes in $[a\ b]$). In other words any $q(x) \in Q_0$ has at most $n - r - 1$ zeroes in $[a\ b] - S$ where:

$$Q_0 = Q_0(E_n^k(r), \bar{\xi}) = \{q(x) \mid q(x) = p_0(x)/\pi(x), p_0(x) \in P_0\} \tag{2.11}$$

and

$$\pi(x) = \prod_{i=1}^{k} (x - \xi_i)^{\mu_i}. \tag{2.12}$$

For a more general discussion of conditions for uniqueness see [16].

3. Alternation Properties

The alternation properties of $f(x) - p(x)$, $p(x) \in P^*(E_n{}^k(r), \bar{\xi})$, as those derived in [11] for matrices containing only H-blocks, are proved straightforwardly if one assumes that Q_0 defined in (2.11) is a Haar space. This is equivalent to the assumption that $E_n{}^k(r)$ is poised after any addition of $n - r$ L-conditions and/or H-conditions (see Definition 1.2). Such matrices are for example the incidence matrices composed of Hermitian blocks and blocks of even length [19]. The following example shows that an L-poised matrix is not necessarily poised after addition of $n - r$ units in the above way.

Example 3.1. Let

$$E_6{}^3(5) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\bar{\xi} = (-1, 0, 1)$. Any $p_0 \in P_0(E_6{}^3(5), \bar{\xi})$ is of the form

$$p_0(x) = C(1 - x)^3 (3x^2 + 9x + 8).$$

Since $3x^2 + 9x + 8 > 0$ for all x, $E_6{}^3(5)$ is L-poised. But by adding to $E_6{}^3(5)$ one H-condition we get:

$$E_6{}^3(6) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

which is nonpoised at $\bar{\xi}$ since

$$p_0(x) \in P_0(E_6{}^3(6), \bar{\xi}).$$

In the following we generalize the alternation properties to the case where $Q_0(E_n{}^k(r), \bar{\xi})$ is not necessarily a Haar space. For this purpose we prove an auxiliary lemma:

LEMMA 3.1. *Let U_m be the linear span of $u_1, \ldots, u_m \in C[a, b]$, and let $T = \{\eta_1, \eta_2, \ldots, \eta_l\}$ be a finite fixed set of points in $[a, b]$. If any nontrivial $u \in U_m$ has at most $m - 1$ zeroes in $[a, b] - T$ and if for any $\eta_i \in T$ there exists at least one $u \in U_m$ such that $u(\eta_i) \neq 0$ then U_m is a Haar space over (a, b).*

Proof. Suppose U_m is not a Haar space, over (a, b). Then there is a $\bar{u} \in U_m$ with m or more zeroes in (a, b); therefore, at least one of them is in T. From the assumption that for any $\eta_i \in T$ there exists a $u \in U_m$ such that $u(\eta_i) \neq 0$, it is easily shown by inductive argument that there exists a $\hat{u} \in U_m$ such that $\hat{u}(\eta_i) \neq 0$, $i = 1, 2, \ldots, l$.

By choosing a proper constant $c > 0$ we show that $\tilde{u} = \bar{u} + c\hat{u} \in U_m$ has at least m zeroes located in $[a, b] - T$.

Let k_1 denote the number of zeroes of $\bar{u}$ that are also zeroes of $\hat{u}$ (obviously not in T). Among the rest of the zeroes of $\bar{u}$ we denote by k_2 the number of those where $\bar{u}$ changes sign (nodal) and by k_3 the number of zeroes with no change of sign (non-nodal). By continuity, for c small enough, $\tilde{u} = \bar{u} + c\hat{u}$ has k_2 nodal zeroes in (a, b) in the neighborhoods of the k_2 nodal zeroes of $\bar{u}$. Let $x_1, \ldots, x_{k_3}$ denote the non-nodal zeroes of $\bar{u}$ (note that $\hat{u}(x_i) \neq 0$, $i = 1, \ldots, k_3$). Then, by taking $-\hat{u}$ if necessary, there are at least $[(k_3 + 1)/2]$ such zeroes where $\hat{u}(x_i)\, \bar{u}(x_i \pm \epsilon) < 0$ for $\epsilon > 0$ small enough. Thus, for $c > 0$ small enough, $\tilde{u}$ changes sign in both $(x_i - \epsilon, x_i)$ and $(x_i, x_i + \epsilon)$ for all those $[(k_3 + 1)/2]$ zeroes, i.e., at least k_3 times. Therefore, $\tilde{u}$ has at least $k_1 + k_2 + k_3 \geqslant m$ zeroes in (a, b), and by taking $c \neq -\bar{u}(\eta_i)/\hat{u}(\eta_i)$, $i = 1, \ldots, l$, we guarantee that $\tilde{u} \neq 0$ on T. Hence $\tilde{u}$ has at least m zeroes in $(a, b) - T$ in contradiction to the assumption of the lemma.

To apply the lemma we introduce:

DEFINITION 3.1. The H-product of an incidence matrix $E_n^k(r)$ at ξ is

$$\tilde{\pi}(x) = \prod_{i=1}^{k} (x - \xi_i)^{\nu_i} \tag{3.1}$$

where $\nu_i \geqslant \mu_i$ $(i = 1, 2, \ldots, k)$ are such that $\tilde{\pi}(x)$ is the polynomial of minimal degree that satisfies the σ homogeneous conditions prescribed by the H-blocks of $E_n^k(r)$ at ξ, and all additional H-conditions that are satisfied by every $p \in P_0(E_n^k(r), \xi)$.

LEMMA 3.2. *Let $E_n^k(r)$ be* L-*poised at $\bar{\xi}$ with respect to $[a, b]$ where $a < \xi_1 < \cdots < \xi_k < b$, and let $\tilde{\pi}(x)$ be its* H-*product there. Then for $\rho > 0$ small enough the space*

$$\tilde{Q}_0 = \tilde{Q}_0(E_n^k(r), \bar{\xi}) = \{\tilde{q}_0(x) \mid \tilde{q}_0(x) = p_0(x)/\tilde{\pi}(x), p_0 \in P_0(E_n^k(r), \bar{\xi})\} \tag{3.2}$$

is a Haar space over $(a - \rho, b + \rho)$.

Proof. By Theorem B and the requirement $\xi_1 \neq a$, $\xi_k \neq b$, $E_n^k(r)$ is also L-poised at $\bar{\xi}$ with respect to $(a - \rho, b + \rho)$ for $\rho > 0$ small enough. Therefore, any $\tilde{q}_0(x) \in \tilde{Q}_0(E_n^k(r), \bar{\xi})$ has at most $n - r - 1$ zeroes in $(a - \rho, b + \rho) - S$ [S is defined in (2.9)]. Moreover, by the definition of $\tilde{\pi}(x)$, for each $\xi_i \in S$ there is a corresponding $\tilde{q}_0 \in \tilde{Q}_0(E_n^k(r), \bar{\xi})$ that does not vanish at ξ_i. Therefore, by Lemma 3.1 $\tilde{Q}_0$ is a Haar space over $(a - \rho, b + \rho)$.

Remark 3.1. By definition, $\tilde{\pi}(x)$, the H-product of an incidence matrix $E_n^k(r)$, is divisible by $\pi(x)$ [defined in (2.12)]. For an L-poised $E_n^k(r)$, $\pi(x) = \tilde{\pi}(x)$ if and only if $Q_0(E_n^k(r), \bar{\xi})$ defined in (3.11) is a Haar space. (See the discussion at the beginning of this section.) The case in [11] falls into this category.

In Example 3.1 the matrix $E_6^3(5)$ is L-poised at $\bar{\xi}$ but is not poised after addition of an H-condition. In this case, $\pi(x) = (x - 1)^2$ but $\tilde{\pi}(x) = (x - 1)^3$.

The result of Lemma 3.2 enables us to prove the necessity of a generalized alternation property under the assumption $\xi_1 \neq a$, $\xi_k \neq b$.

THEOREM 3.1. *Let $p \in P(E_n^k(r), \bar{\xi})$ be a* pba *to $f \in C[a, b]$, $\bar{n}$, $\bar{r}$ as defined in Theorem 2.2. Then there exist $\bar{n} + 1 - \bar{r}$ consecutive points $x_i \in [a, b]$ such that:*

$$|f(x_i) - p(x_i)| = \|f - p\| \qquad i = 1, 2,..., \bar{n} + 1 - \bar{r} \tag{3.3}$$

and

$$\operatorname{sign}\{[f(x_i) - p(x_i)]\,\tilde{\pi}(x_i)\} = (-1)^{i-1} \operatorname{sign}\{[f(x_1) - p(x_1)]\,\tilde{\pi}(x_1)\} \qquad i = 1, 2,..., \bar{n} + 1 - \bar{r} \tag{3.4}$$

where $\tilde{\pi}(x)$ is the H-*product of $E_{\bar{n}}^k(\bar{r})$ at $\bar{\xi}$.*

Proof. The method of proof is the same as the one given by Davis [3] in the case of approximation without side conditions. Divide $[a, b]$ into consecutive closed intervals in which the maximal change of the error function $\epsilon(x) = f(x) - p(x)$ is $< \epsilon/2$ where $\epsilon = \|\epsilon(x)\|$. Any interval where $|\epsilon(x)| = \epsilon$ at least once does not contain a point of S since $|\epsilon(x)| > \epsilon/2$ on it while on S, $\epsilon(x) = 0$. We group these intervals into consecutive groups, starting a new group only when there is a change of sign of $\epsilon(x)\,\tilde{\pi}(x)$. From each group we take one point x_i, $|\epsilon(x_i)| = \epsilon$, $i = 1, 2,..., l$. The claim of the theorem holds trivially if $\bar{n} = \bar{r}$ or if $l \geqslant \bar{n} + 1 - \bar{r}$.

Suppose to the contrary that $1 \leqslant l \leqslant \bar{n} - \bar{r}$, $\bar{n} > \bar{r}$. By the above construction there is an open interval between any two consecutive groups. From each such interval we choose one point y_i, $i = 1, 2,..., l - 1$ such that $y_i \notin S$.

Since by Lemma 3.2 $\tilde{Q}_0(E_{\bar{n}}^k(\bar{r}), \bar{\xi})$ is a Haar space over (a, b) it is possible to construct a polynomial $q_0 \in \tilde{Q}_0$ that has $l - 1$ simple zeroes at $y_1, \ldots, y_{l-1}$ and is different from zero elsewhere in (a, b) (see [8, Theorem 5.2]). Let $\tilde{q}_0(x)\, \tilde{\pi}(x) = p_0(x)$. Then by the choice of $y_1, \ldots, y_{l-1}$, the product:

$$\epsilon(x)\, p_0(x) = \epsilon(x)\, \tilde{\pi}(x)\, \tilde{q}_0(x)$$

is of constant sign on $A(f, p)$ [defined in (2.2)], in contradiction to Theorem 2.1. Therefore, $l \geqslant \bar{n} + 1 - \bar{r}$.

Remark 3.2. As indicated in Remark 2.2, the above choice of $\bar{n}$ gives the maximal number of alternatations. Moreover, by taking a partial L-poised matrix with a smaller $\bar{n}$ we may get a different kind of alternation, since the H-product of a smaller partial matrix is not necessarily identical to the one corresponding to the maximal $\bar{n}$.

Note also that $\tilde{\pi}(x)$ used in Theorem 3.1 is divisible by $\pi(x)$ defined in (2.12) with respect to the original matrix $E_n{}^k(r)$. Example 2.3 and the following example show that Theorem 3.1 is sharp.

EXAMPLE 3.2. Let $E_3{}^1(1) = (010)$, $\bar{\xi} = (0)$, $[a, b] = [-1, 1]$, $f(x) = \sin(\pi/2)x$, $b_{11} = 0$ and

$$P(E_3{}^1(1), \bar{\xi}) = P_0(E_3{}^1(1), \bar{\xi}) = \{p(x) \mid p \in \pi_2 \,.\, p'(0) = 0\}.$$

Each $p \in P(E_3{}^1(1), \bar{\xi})$ is of the form $p(x) = ax^2 + b$. Since $p(-x) = p(x)$ it is easily seen that any pba must satisfy $p(-1) = p(1) = 0$. Here $\bar{n} = 2$, $\bar{r} = 1$, $\bar{n} + 1 - \bar{r} = 2$ and $\tilde{\pi}(x) = \pi(x) = 1$, so that A^* contains only two points, -1 and $+1$, with the Tchebycheff alternation.

If $E_n{}^k(r)$ is L-poised at $\bar{\xi}$, then the alternation property of Theorem 3.1 holds for $n + 1 - r$ points with $\tilde{\pi}(x)$ the H-product of $E_n{}^k(r)$. In this case it is also a sufficient condition for $p(x)$ to be the unique pba, as is proven in the next theorem.

THEOREM 3.2. *Let $E_n{}^k(r)$ be* L-*poised at $\bar{\xi}$ and suppose there is a polynomial $p(x) \in P(E_n{}^k(r), \bar{\xi})$ for which* (3.3) *and* (3.4) *hold with $\bar{n} = n$ $\bar{r} = r$. Then $p(x)$ is the* pba *to f from $P(E_n{}^k(r), \bar{\xi})$.*

Proof. By Corollary 2.2 there is a unique pba to f from P. Suppose $p(x)$ is not that pba then there exists $\hat{p}(x) \in P$ for which

$$\| f(x) - \hat{p}(x) \| = \delta < \epsilon \qquad \epsilon = \| f(x) - p(x) \|.$$

Denote by $\delta_i = f(x_i) - \hat{p}(x_i)$ $\epsilon_i = f(x_i) - p(x_i)$ where x_i $i = 1 \ldots n + 1 - r$ are the points in (3.3) and (3.4). Then

$$h(x) = p(x) - \hat{p}(x) \tag{3.5}$$

satisfies

$$h(x_i) = -\epsilon_i + \delta_i \qquad i = 1,\dots, n+1-r.$$

Since $|\delta_i| \leqslant \delta < \epsilon = |\epsilon_i|$ we conclude that

$$\operatorname{sign} h(x_i) = \operatorname{sign}(-\epsilon_i) \qquad i = 1,\dots, n+1-r. \tag{3.6}$$

By (3.5) $h(x) \in P_0(E_n{}^k(r), \bar{\xi})$ and can be factorized into $\tilde{\pi}(x)\,\tilde{q}_0(x)$ where $\tilde{\pi}(x)$ is the H-product of $E_n{}^k(r)$ and $\tilde{q}_0(x) \in \tilde{Q}_0(E_n{}^k(r), \bar{\xi})$. By (3.6)

$$\operatorname{sign} \tilde{\pi}(x_i)\,\tilde{q}_0(x_i) = \operatorname{sign}(-\epsilon_i) = -\operatorname{sign}[f(x_i) - p(x_i)] \qquad i = 1,\dots, n+1-r. \tag{3.7}$$

However, using assumption (3.4) of the theorem we get:

$$\operatorname{sign} \tilde{q}_0(x_i) = (-1)^i \operatorname{sign}\{[f(x_1) - p(x_1)]\,\tilde{\pi}(x_1)\} \qquad i = 1,\dots, n+1-r$$

i.e., $\tilde{q}_0(x)$ has at least $n - r$ sign changes in (a, b). But since by Lemma 3.2 $\tilde{Q}_0(E_n{}^k(r), \bar{\xi})$ is a Haar space, we get a contradiction.

The ideas of this paper are applied to the problem of monotone approximation [13] and "restricted derivatives" [17] in a later work [10].

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